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The revision and extension of the R_{MS} ring for time delay systems

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Abstract. This paper is aimed at reviewing the ring of retarded quasipolynomial meromorphic functions (R_{MS}) that was recently introduced as a convenient control design tool for linear, time-invariant time delay systems (TDS). It has been found by the authors that the original definition does not constitute a ring and has some essential deficiencies, and hence it could not be used for an algebraic control design without a thorough reformulation which i.a. extends the usability to neutral TDS and to those with distributed delays. This contribution summarizes the original definition of R_{MS}, simply highlights its deficiencies via examples, and suggests a possible new extended definition. Hence, the new ring of quasipolynomial meromorphic functions (R_{QM}) is established to avoid confusion. The paper also investigates and introduces selected algebraic properties supported by some illustrative examples and concisely outlines its use in controller design.

Key words: time delay systems, ring, Bézout domain, coprime factorization.

1. Introduction

We consider a general single-input single-output (SISO) linear time-invariant time delay system (TDS) as

$$\begin{aligned}\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} &= \sum_{i=1}^{\nu_H}\mathbf{H}_i\frac{\mathrm{d}\mathbf{x}(t-\vartheta_i)}{\mathrm{d}t}+\sum_{i=1}^{\nu_A}\mathbf{H}_i\mathbf{A}_i\mathbf{x}(t-\vartheta_i)+\mathbf{A}_0\mathbf{x}(t)\\ &+\sum_{i=1}^{\nu_B}\mathbf{b}_0^T u(t-\vartheta_i)+\mathbf{b}_0^T u(t)\\ &+\int_0^L\left(\widetilde{\mathbf{A}}(\tau)\mathbf{x}(t-\tau)+\mathbf{b}^T(\tau)u(t-\tau)\right)\mathrm{d}\tau \\ y(t) &= \mathbf{c}\mathbf{x}(t)+\int_0^L\widetilde{\mathbf{c}}(\tau)\mathbf{x}(t-\tau)\mathrm{d}\tau\end{aligned}\tag{1}$$

where $\mathbf{x}\in\mathbb{R}^n$ is the vector of state variables, $u, y\in\mathbb{R}$ stand for the input and the output, respectively, $\mathbf{A}_i$, $\mathbf{A}_i$, $\widetilde{\mathbf{A}}(\tau)$, $\mathbf{b}_i$, $\widetilde{\mathbf{b}}(\tau)$, $\mathbf{c}$, $\widetilde{\mathbf{c}}(\tau)$, $\mathbf{H}_i$ represent vectors and matrices of compatible dimensions, $0<\vartheta_i\le L$ are lumped delays and the convolution integrals express distributed delays [1, 2]. Commensurate delays are integer multiples of some base delay. If $\mathbf{H}_i\neq\mathbf{0}$ for any $i=1, 2, ..., \nu_H$, model (1) is called neutral; contrariwise, the so-called retarded model is obtained. Delays can inherently act throughout various human activities [3, 4].

Regarding ring models rising from (1) or its part, from the historical point of view, the general concept of systems over rings [5] was first applied to infinite-dimensional linear systems by Kamen [6] via rings of distributions. Ring models for TDS with lumped delays were published in [7]. Sontag [8] introduced the ring of polynomials in delayed operators for both the commensurate and non-commensurate delays. These approaches utilize the state space domain and arise from the two-dimensional (2D) conception of algebraically independent derivative operators and delays. The existence and constructing of stabilizing finite-dimensional compensators for non-commensurate TDS in the 2D polynomial ring was discussed in [9]. A general mathematical setting for the stabilization and control of TDS by the generalization of algebraic methods in 2D, with the ring of lumped and distributed delays and with the complexity of generalness, was first introduced in [10]. It is worth highlighting that quasipolynomials defined in this sense, regardless whether in 2D or purely in the Laplace transform operator s, are connected with commensurate delays. It is, however, rather restrictive for real applications to be focused on commensurate delays only since delays are naturally real-valued with arbitrary mutual ratios. Brethé and Loiseau [11] pointed out that the use of quasipolynomials in s does not permit to effectively handle some stabilization and control tasks and suggested the ring of pseudopolynomials. Linear algebra for commutative rings was summarized in [12]. A very useful overview including also the algebraic point of view of general systems with distribution was provided in [13] where notions such as the properness, stability, and minimum-phase systems, different from the finite-dimensional case, were given to the reader. Also, note that the so-called σ-algebra was used to investigate the complete controllability of stochastic models with finite distributed delays in [14].

An effective way of dealing with control and stabilization tasks may consist of the introduction of the fractional representation approach [15–17] that can be extended from rational transfer functions to TDS in various algebras [18], and is usually based on the solution of the Bézout identity [19]. One may take a rational approximation of exponential terms, which brings a loss of system dynamics information that can disproportionately increase the model order [20]. However, in the case of non-approximated transfer functions, there are many possibilities that might be confused with each other, such as the set of stable and proper retarded quasipolynomial meromor-

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phic functions (R_{MS}) [21, 22], the Callier-Desoer class [23], the BBZ ring [24], the H_∞ set [16, 25], the algebra A [26], etc. The mutual inclusions and relations analysis and the determination of the affiliation with the particular algebra pose a difficult task, mainly due to specific stability properties of neutral TDS [2, 27].

Within our research framework, we concentrate on the fractional representation in the R_{MS} ring standing somewhere between H_∞, algebra A, and BIBO (bounded-input bounded-output) stable fraction. It reflects the fact that that the z-transform and the Laplace transform operators are not independent from the functional point of view, it does not require any rational approximation, it is not limited to commensurate delays, and it is simple enough and suitable to cope with practical stabilization and control tasks [28, 29].

Although the original definition of the ring [21] is sufficient to be used for control tasks in the overwhelming majority of cases, it suffers from some drawbacks which make the R_{MS} structure inapplicable for many controlled plant models, such as those with distributed or neutral delays, as we have found during the work with R_{MS}. In particular, it does not constitute a ring, which is an essential problem. Thus, the aim of this paper is to point out basic deficiencies in the definition, revise the concept of R_{MS} and extend it; hence, the ring R_{QM} is established. Note that a preliminary attempt to analyze selected imperfections in the original definition was already made in [30], where, however, some ideas were not presented flawlessly and completely. Thus, the presented contribution provides in some sense completion and adjustment of our observations. The ideas and statements are illustrated by examples introduced throughout the paper to illuminate them for the reader who is supposed to be acquainted with basic algebraic notions, such as a ring or a field, and with the essentials of complex analysis. If necessary, uncommon notions and statements are provided here. It is worth noting that although the particular controller design in R_{QM} is not the main message of this paper, an illustrative example is given as well. The reader is referred for details to the analogous topic solved for R_{QM} e.g. in [29].

The paper is organized as follows. An overview of stability notions for system (1), elementary general algebraic terms and properties, basics of complex analysis and a summary of the original definition of R_{MS} are provided in the preliminary Section 2, followed by the attention drawn to highlight its deficiencies given via examples. The revision giving rise to the definition of R_{QM} and the consequential discussion are the content of Section 3. In Section 4, selected algebraic and functional properties of the revised ring definition are introduced. The usability of R_{QM} for control design is outlined in Section 5 via a concise example. Finally, Section 6 concludes the paper.

Thorough the paper, $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ denote the set of complex numbers, real numbers, and non-negative integers, respectively. We use $\mathcal{L}(\cdot)$ for the Laplace transform of $(\cdot)$. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ denote, respectively, the real part and imaginary part of s, $\mathbb{C}^- := \{s \in \mathbb{C} \,|\, \mathrm{Re}(s) < 0\}$, $\mathbb{C}_0^+ = \mathbb{C} \backslash \mathbb{C}^-$, the set of polynomials is denoted as $\mathbb{R}[s]$. It holds that $(\cdot) \in H_\infty \Leftrightarrow \|(\cdot)\|_\infty := \sup_{\mathrm{Re}(s) \geq 0} |(\cdot)| < \infty$.

2. Preliminaries

The direct use of the Laplace transform to (1) yields the transfer function $G(s) = b(s)/a(s)$, where $a(s)$, $b(s)$ are quasipolynomials of the general form

$$q(s) = s^\nu + \sum_{i=0}^{\nu} \sum_{j=1}^{h_i} q_{ij} s^i \exp(-s\tau_{ij}), \tau_{ij} \geq 0, q_{ij} \in \mathbb{R}, \quad (2)$$

where if $\exists k : \tau_{\nu k} \neq 0$, then $q_{\nu k} = 0$ holds for the retarded one, and $\exists k$, such that $\tau_{\nu k} \neq 0$, $q_{\nu k} \neq 0$ holds in the neutral case.

2.1. TDS stability. Let us concisely introduce basic notions regarding TDS stability, useful in the text hereinafter.

Exponential stability simply agrees with the finite-dimensional case, i.e. all system poles satisfying $1/G(s) = 0$ have (strictly) negative real parts. A system is said to be H_∞ stable if $G(s) \in H_\infty$ (i.e. the function is analytic and bounded in $\mathbb{C}_0^+$) [18]. Particularly for neutral TDS, a transfer function having no pole in $\mathbb{C}_0^+$ but an infinite sequence of poles with real parts converging to zero can be H_∞ unstable due to unbounded gain at the imaginary axis. Further, a system is BIBO stable if a bounded input $|u(t)| < M_1$, $M_1 > 0$ implies a bounded output $|y(t)| < M_2$, $M_2 > 0$. The decision about BIBO stability is usually more difficult to analyze, and it holds that BIBO stability implies H_∞ stability [18, 26].

Regarding very specific stability notions for TDS, formal stability (formulated primarily in the state space [31]) can be given in the parlance of the Laplace transfer function as follows: A neutral TDS is formally stable if it has only a finite number of poles in $\mathbb{C}^-$, i.e. the rightmost vertical strip of poles of a neutral system does not reach or cross the imaginary axis. However, there is no simple rule how to ascertain formal stability from the transfer function. Therefore, let us mention a similar, yet a rather stronger stability notion – strong stability. This type of neutral TDS stability means that the vertical strip remains in $\mathbb{C}^-$ when subjected to small variations in delays, i.e. the system remains formally stable. E.g. in [32], a simple strong stability criterion was provided as

$$\sum_{j=1}^{h_\nu} |a_{\nu j}| < 1. \quad (3)$$

From the above introduced stability notions it is evident that the strong stability condition (3) implies the formal stability and it can be used as a sufficient formal stability test (with some conservativeness).

2.2. Algebraic and complex analysis notions, operations and properties. The reader is supposed to be acquainted with elementary algebraic notions (such as a ring, a field, an integral domain, an irreducible, and a prime element of the commutative ring), algebraic operations and features (the divisibility, the coprimeness, the associativity), and terms from complex analysis (poles of a meromorphic function, etc.). We add some less known ones, yet necessary for the further text, as well as selected results [33].

A ring R in which every nonzero noninvertible $a \in R$ can uniquely be decomposed in a product of a finite number of irreducible or prime elements (except for the ordering and associativity) is called a unique factorization ring (UFR). If, moreover, R is an integral domain, the ring constitutes *a* unique factorization domain (UFD).

An ideal I of the ring R is a subset of R with the following properties: for every $a, b \in I$ it holds that $a + b \in I$, and for each $a \in I$ and $r \in R$ it holds $a \cdot r \in I$. Let be given $M = \{a_1, a_2, \ldots, a_n\} \subseteq R$; an intersection of all ideals of R containing M is called an ideal generated by M. Ideals of the form $aR = \{a \cdot r | r \in R\}$, i.e. those generated by the single element a, are called principal. If every ideal of an integral domain is principal, a so-called principal ideal domain (ID) is obtained. In a Bézout domain, every finitely generated ideal is principal.

Proposition 2.1. Every principal ID is UFD. The converse is not true in general [33].

Definition 2.1. $T_1(s), T_2(s) \in H_\infty$ form a Bézout (coprime) factorization iff [13], [19]:

$$\inf_{\mathrm{Re}(s) \geq 0} \left(|T_1(s)| + |T_2(s)| \right) > 0. \tag{4}$$

Proposition 2.2. If a neutral TDS governed by the transfer function $G(s) = B(s)/A(s)$, $A(s), B(s) \in H_\infty$ is BIBO stabilizable, then it holds that it admits a Bézout factorization over H_∞, that there exists a coprime pair $X(s), Y(s) \in H_\infty$ such that

$$A(s)X(s) + B(s)Y(s) = 1, \tag{5}$$

and that any coprime factorization $G(s) = B(s)/A(s)$ is Bézout [13, 19].

Rephrasing Proposition 2.2, if (4) does not hold for a coprime factorization, the factorization is not Bézout and thus the system is not BIBO stabilizable. For an example of a coprime factorization not being Bézout, the reader is referred to [26] or Example 3.1 in this paper.

Proposition 2.3. In a Bézout domain R, for every pair there exists the greatest common divisor (GCD) which satisfies the linear Diophantine equation [33]:

$$\begin{aligned} & a \cdot x + b \cdot y = \mathrm{GCD}(a, b) \\ \Leftrightarrow & \frac{a}{\mathrm{GCD}(a,b)} \cdot x + \frac{b}{\mathrm{GCD}(a,b)} \cdot y = 1, x, y \in R. \end{aligned} \tag{6}$$

The extended (generalized) Euclidean algorithm solving (6) – and also (5) – for a general Bézout ring R can be descried as follows: Set initial reminders as $r_1 = a$ and $r_2 = b$. In the ith iteration it holds that $r_i = r_{i-2} - \lfloor q_i \rfloor \cdot r_{i-1}$, $r_{i-2} \geq r_{i-1} \geq r_i$, $i = 3, 4, \ldots, n$, where q_i is the quotient. It is always possible to write the identity $r_i = a \cdot x_i + b \cdot y_i$ for some $x_i, y_i \in R$. The eventual d then equals the last nonzero remainder, $r_n \neq 0$, $n < \infty$.

The whole procedure can be expressed in a table (matrix) form as follows:

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}\right] \sim \begin{array}{c}\text{elementary}\\ \text{matrix}\\ \text{operations}\end{array} \sim \left[\begin{array}{cc|c} v & t & 0 \\ x & y & d \end{array}\right], \tag{7}$$

and then the result is determined by two equations $b \cdot v + b \cdot t = 0$, $a \cdot x + b \cdot y = d$.

In the case when the task is to solve (6) for any fixed $c \in R$ on the right-hand side instead of $d = \mathrm{GDC}(a, b)$, it is possible to use the extended Euclidean algorithm again (if a solution exists) in the following two possibilities. Either scheme (7) is used for c instead of d (generally, it is not necessary to achieve the zero entry on the upper right matrix corner), or $a\widetilde{x} + b \cdot \widetilde{y} = c$, where:

$$\widetilde{x} = x\frac{c}{d}, \ \widetilde{y} = y\frac{c}{d}. \tag{8}$$

Note that a (particular) solution of (6), x_0, y_0, can be parameterized as $x = x_0 \pm t(b/d)$, $y = y_0 \mp t(a/d)$ for any $t \in R$.

Definition 2.2. A partially ordered set (poset) is an ordered pair $P = (S, \preceq)$ where S stands for the ground set of P and $\preceq$ expresses the partial order of P. For any $a, b, c \in S$ it holds that: $a \preceq a$; if $a \preceq b$ and $b \preceq a$, then $a \equiv b$; $a \preceq b$ and $b \preceq c$ implies $a \preceq c$.

2.3. R_{MS} definition and its deficiencies.

Definition 2.3. (R_{MS} ring) [21] $T(s) = n(s)/d(s) \in R_{MS}$, where $n(s)$ and $d(s)$ are retarded quasipolynomials with $\deg_s d(s) = v_d$, and $n(s)$ factorizable as $n(s) = \tilde{n}(s)\exp(-\tau s)$, where $\tilde{n}(s)$ is a retarded quasipolynomial with $\deg_s \tilde{n}(s) = v_n$, $\tau > 0$. Moreover, $d(s)$ is stable in the sense that it has no zero $s_i \in \mathbb{C}_0^+$, and the ratio is proper in the sense $v_n \leq v_d$.

A deeper insight into the formulation of Definition 2.3 brings some imperfections into the light. First, the condition $\tau > 0$ is undue restrictive or more probably a misprint, hence, the inequality $\tau \geq 0$ would be more natural instead. The cardinal drawback exists in the finding that the defined algebraic set does not constitute a ring, which is shown in the following example.

Example 2.1. Consider $T_1(s) = (s\exp(-2s))/(s+1)$ and $T_2(s) = ((s+2)\exp(-s))/(s+1)$ satisfying Definition 2.3. The sum $T_1(s) + T_2(s)$, however, does not meet the definition, since $\tilde{n}(s) = s(1 + \exp(-s)) + 2\exp(-s)$ is a neutral quasipolynomial. Although the R_{MS} structure has been introduced to pursue retarded TDS, this example indicates that it is necessary to include neutral terms in the definition.

Example 2.2. Another drawback comes from the requirement of a stable denominator. Consider the finite convolution expressing distributed delays as $y(t) = \int_0^1 \exp(\tau)u(t-\tau)d\tau$ giving rise to the transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1-\exp(1)\exp(-s)}{s-1}. \tag{9}$$

The function includes the denominator with root $s_0 = 1$ while the whole system is stable. Thus, an element of the ring can include a removable singularity in $\mathbb{C}_0^+$ not being poles.

3. The extension of the conception *R*QM

Based on the examples above, we propose an extension of the conception of R_{MS}, called the R_{QM} ring, and provide the reader with a discussion on this definition.

Definition 3.1. (R_{QM} ring) $T(s) = n(s)/d(s) \in R_{QM}$, where $n(s)$, $d(s)$ are neutral quasipolynomials (in general) and $n(s)$ is factorizable as $n(s) = \tilde{n}(s)\exp(-\tau s)$, $\tau \geq 0$. Moreover, $T(s)$ is formally stable and $T(s) \in H_\infty$.

Discussed now will be some issues formulated within the definition. It is habitual that $T(s)$ is proper; the condition

$$\sup_{\mathrm{Re}(s)>0, |s|\geq R} |T(s)| > \infty \tag{10}$$

for some $R > 0$, expresses that $T(s)$ is proper in a more general way than the usual formulation via the highest s-powers [26]. However, by comparison of the H_∞ set with (10), it is evident that H_∞ implies (10), since it is sufficient to take any positive R.

Formal and H_∞ stability will be discussed in more detail. Loiseau et al. [31] stated that a system which is not formally stable is not H_∞ and hence, not BIBO stable and stabilizable. Nevertheless, Partington and Bonnet [26] revised this statement and showed that a formally unstable TDS can be H_∞ and BIBO stable but not stabilizable, as seen in the following example.

Example 3.1. Let be given three different neutral delayed systems governed by transfer functions:

$$G_1(s) = \frac{1}{s + s\exp(-s) + 1}, G_2(s) = \frac{G_1(s)}{s+1} = \frac{b(s)}{a(s)}, \quad G_3(s) = \frac{G_1(s)}{(s+1)^4} \tag{11}$$

It can be verified that all the functions in (11) have poles located in $\mathbb{C}^-$ except for poles s_i with $\mathrm{Im}(s_i) \to \infty$, where the asymptotic vertical chain of roots tends to the imaginary axis. However, they cannot be considered as asymptotically (exponentially) stable since there is no $\alpha > 0$ satisfying $\mathrm{Re}(s_i) \leq -\alpha$ for all s_i. These systems are neither strongly nor formally stable; condition (3) is not valid in any case. Numerical tests show that $\|G_1\|_\infty = \infty$, $\|G_2\|_\infty = 2$, $\|G_3\|_\infty = 1$, hence $G_1 \notin H_\infty$, $G_2, G_3 \in H_\infty$. Moreover, as proved in [26], G_1 and G_2 are not BIBO stable but G_3 is BIBO stable. It means that formal instability does not automatically imply H_∞ or BIBO instability.

In addition, the requirement of formal (or strong) stability is not desirable only from the practical point of view, resulting from the stability definitions, but also for algebraic reasons. Consider a coprime factorization of system $G_2(s)$ from (11) over H_∞ as

$$G_2(s) = B(s)/A(s), B(s) = b(s)/(s+2)^2, A(s) = a(s)/(s+2)^2$$

The factorization is coprime in the sense that there is no nontrivial (nonunit) common factor of $A(s), B(s) \in H_\infty$, yet it is not Bézout (in infinity). As stated above, G_2 is formally unstable, but from H_∞. However, one can verify that $A^{-1}(s) \notin H_\infty$. Hence, although there is no noninvertible common factor of both to be cancelled, the fraction $B(s)/A(s) = B(s)A^{-1}(s)$ seems to implicitly perform such a cancelation. This yields somewhat of a mismatch in the ring definition since there is not an unambiguous answer whether $A(s)$ is a unit (i.e. an invertible element) or not. Obviously, if $A(s)$, $B(s)$ were not coprime, it would not pose a problem, since the particular nonunit factor would be canceled by the division $A(s), B(s) \in H_\infty$. Moreover, if G_2 was formally stable, the inversion $A^{-1}(s)$ would also be included in H_∞ and hence, $A(s)$ would be considered invertible. To sum up, the set H_∞ itself cannot be a sufficient candidate for the R_{QM} ring due to the existence of formally unstable neutral TDS, and formal (or strong) stability is desirable to be required in the definition.

Last but not least, the question is why cannot the strong stability though a simple test (3) be included in the definition of R_{QM}. Consider the strongly stable denominator quasipolynomial $a(s) = \left(1 + 0.9\exp(-s)\right)s + 1$ of some $T(s) \in H_\infty$. Clearly the square $\bar{a}(s) = a^2(s)$ has $|\bar{a}_{11}| + |\bar{a}_{21}| = 1.8 + 0.81 = 2.61 > 1$, i.e. $T^2(s)$ is strongly unstable but formally stable since $T(s)$ and $T^2(s)$ own the same spectrum, except for poles multiplicities (it is generalized in Proposition 3.1, the proof of which is introduced in the Appendix).

Proposition 3.1. Given two formally stable neutral terms $T_1(s), T_2(s) \in H_\infty$, the formal stability property over H_∞ is closed under addition and multiplication.

The primary task in the control design is to stabilize the control feedback system, therefore it is desirable to get the element of R_{QM} from H_∞ and, from the examples above, to ensure that it is, in addition, formally stable. Moreover, if the neutral system is of a nonzero relative order, one may wish for it to be formally stable as well as to avoid a formally unstable numerator quasipolynomial in $T(s) \in R_{QM}$, and thus to have a Bézout coprime stabilizing pair, which, however, is not possible in all cases [26]. Note that the relative order of neutral system $G(s) = b(s)/a(s)$ equals $\deg_s a(s) - \deg_s b(s)$.

4. Some *R*QM properties and operations

Adopting the concept of the R_{QM} ring established in Definition 3.1 we are going to derive and provide some elementary algebraic properties of the ring, along with the presentation of operation over the ring useful for control design, in the following subsections.

4.1. Algebraic properties.

Lemma 4.1. The R_{QM} set constitutes a commutative ring.

The proof is given to the reader in the Appendix.

Lemma 4.2. An element $T(s) \in R_{QM}$ is a unit (i.e. an invertible element) iff $T(s)$ has zero relative order, has no zero z_0 such that $\operatorname{Re} z_0 \geq 0$, and has a formally stable numerator.

Proof. The proof is evident since the required relative order ensures the properness of $T^{-1}(s)$, stable zeros give rise to stable poles of the inversion, and a formally stable numerator transforms into a formally stable denominator. The reader can easily deduce that both the required implications are clear.

Lemma 4.3. An element $T(s) \in R_{QM}$ is irreducible iff its numerator is formally stable and

$$O_R + N_U \leq 1, \tag{12}$$

where $O_R \geq 0$ stands for the relative order of $T(s)$ and $N_U \geq 0$ is the number of real zeros $s_{U,i}$, $i = 1, 2, \ldots, N_U$ or conjugate pairs $s_{U,i}$, $\overline{s}_{U,i}$, $i = 1, 2, \ldots, N_U$ with $\operatorname{Re}(s_{U,i}) \geq 0$ and $\operatorname{Re}(\overline{s}_{U,i}) \geq 0$ (excluding infinity) of $T(s)$, respectively.

Again, the proof can be found in the Appendix.

Lemma 4.4. The R_{QM} ring constitutes an integral domain; however, it is not a UFD.

See Appendix for proof.

Lemma 4.5. The R_{QM} ring does not constitute a principal ID.

The proof is clearly seen from Lemma 4.4 and Proposition 2.1.

Lemma 4.6. The R_{QM} ring does not constitute a Bézout domain.

A proof that can be done using Propositions 2.2 and 2.3 is omitted since the lemma is not constructive for the practice.

4.2. Algebraic operations. Dealing with algebraic control design for a TDS, a (Bézout) coprime factorization, by which the transfer function is decomposed into a coprime (or relatively prime) pair of ring elements (see e.g. Example 3.1), and the solution of the Bézout identity (5) or (6), leading to a stable feedback system, are one of the mostly used operations. Hence, let us briefly present some details about these techniques over the R_{QM} ring.

The crucial problem is to decide whether for a particular $T_1(s)$, $T_2(s) \in R_{QM}$ it holds true that $T_1(s)$ divides $T_2(s)$, i.e. if $T_2(s)/T_1(s) \in R_{QM}$.

Lemma 4.7. Any $T_1(s) = n_1(s)/d_1(s) \in R_{QM}$ divides $T_2(s) = n_2(s)/d_2(s) \in R_{QM}$, or $T_1(s)|T_2(s)$, iff all finite zeros $z_i \in \mathbb{C}_0^+$ of $T_1(s)$ are those of $T_2(s)$, the relative order of $T_1(s)$ is less or equal to the relative order of $T_2(s)$, and all formally unstable factors of the numerator of $T_1(s)$ are those of $T_2(s)$.

The proof of Lemma 4.7 is evident and therefore can be omitted.

Remark 4.1. Dealing with TDS brings about an interesting feature which is unparalleled to a finite-dimensional case, where if $T(s) = n(s)/d(s)$, $n(s), d(s) \in \mathbb{R}[s]$, and $\deg d(s) > \deg n(s)$, there exists at least one zero $|z_0| \to \infty$. This, however, is not true for quasipolynomials and their fractions. Let a term $T(s) = n(s)/d(s) \in R_{QM}$ with $\deg_s d(s) > \deg_s n(s)$ and no common roots of the numerator and the denominator be considered. Choose $\varepsilon > 0$ and consider a chain of poles s_i of $T(s)$, where $\exists n, \forall i > n: |s_i| > 1/\varepsilon$. Then, the limit $\lim_{|s|>1/\varepsilon, \varepsilon \to 0} T(s) = \lim_{|\sigma|<\varepsilon, \varepsilon \to 0} T(1/\sigma) = \lim_{|\sigma|<\varepsilon, \varepsilon \to 0} F(\sigma)$ does not exist, since whenever the zero of $F(s)$ is approached on the disk $D(0, \varepsilon)$ outside the points $\sigma_i = s_i^{-1}$, the limit goes to zero, whereas the limit reaches infinity exactly at these points. It means that in $|s| \to \infty$ function $T(s)$ has the so-called essential singularities and there is no zero in infinity of the function. Therefore, the formulation including the statement about zeros in infinity, habitual for a finite-dimensional case [20], cannot be used in Lemma 4.7.

Regarding the coprime factorization, recall that problems appear when dealing with neutral TDS or with those including distributed delays. An example of coprime, yet not Bézout factorization of a formally unstable neutral TDS was demonstrated in Example 3.1 and e.g. in [26]. The task is to obtain a coprime pair for a (formally stable) TDS.

Lemma 4.8. Consider a TDS governed by the transfer function $G(s) = b(s)/a(s)$ where $a(s)$, $b(s)$ are quasipolynomials with $\deg_s b(s) \leq \deg_s a(s)$. The system has a coprime factorization

$$G(s) = \frac{b(s)}{a(s)} = \frac{\dfrac{b(s)}{m(s)}}{\dfrac{a(s)}{m(s)}} = \frac{B(s)}{A(s)} \tag{13}$$

over R_{QM}, i.e. $A(s), B(s) \in R_{MS}$ are coprime or relatively prime iff $\deg_s m(s) = \deg_s a(s)$ and all common roots $s_i : a(s_i) = b(s_i) = 0$, $s_i \in \mathbb{C}_0^+$ are the only unstable roots of the formally stable (quasi) polynomial $m(s)$.

For a proof, the reader is referred to Appendix. Note that the coprime factorization, according to Lemma 4.8, does not imply that it is a Bézout one, and also that even if it exists, a noninvertible $T(s) \in R_{QM}$ for $A(s) = \widetilde{A}(s)T(s)$ $A(s)$ can still be irreducible since $T(s)$ might be associated with $A(s)$ and $\widetilde{A}(s)$ might be a unit. Therefore, Lemma 4.3 cannot be used directly in the proof.

Example 4.1. Consider a simple system with distributed delays governed by the transfer function (9) and suggest a coprime factorization. In this case, the common denominator (quasi)polynomial $m(s)$ cannot be stable, since it would lead to scheme (22). Therefore, $m(s)$ must include all common zeros s_i with $\operatorname{Re} s_i \geq 0$. Thus, the coprime factorization should read

$$G(s) = \frac{1 - \exp(1)\exp(-s)}{s-1} = \frac{\dfrac{1 - \exp(1)\exp(-s)}{s-1}}{\dfrac{s-1}{s-1}} = \frac{B(s)}{A(s)} \quad (14)$$

Remark 4.2. The concept of the R_{QM} ring resulting in Lemma 4.8 excludes the existence of a coprime factorization (or R_{QM} elements) for some mathematical "monsters". For instance, assume hypothetically the following transfer function $G(s) = b(s)/a(s) = \big(1 - \exp(-3s)\big)/\big(1 - \exp(-2s)\big)$, where both the neutral quasipolynomials $a(s)$, $b(s)$ are formally unstable with their roots $s_{a,k,k+1} = \pm k\pi j$ and $s_{b,k,k+1} = \pm 2/3k\pi j$, $k \in \mathbb{N}$, respectively. Thus, there exists an infinite number of different unstable roots of $A(s)$ and $B(s)$. However, there are also infinitely many common unstable roots $s_{k,k+1} = \operatorname{LCM}(s_{A,k,k+1}, s_{B,k,k+1}) = \pm 2k\pi$, where LCM(.,.) denotes the least common multiple. The monic quasipolynomial of the zero-degree owning exactly roots $s_{k,k+1}$ reads $m(s) = 1 - \exp(-s)$. The coprime (even Bézout coprime) factorization would be $A(s) = a(s)/m(s)$, $B(s) = b(s)/m(s)$. Obviously, $A(s), B(s) \notin R_{QM}$ according to Definition 2.3, due to the formally unstable denominator. However, both expressions have no pole in $\mathbb{C}_0^+$ and it is possible to establish a ring concept that would accept such terms and factorizations. Nevertheless, it is arguable whether this endeavour would be useful and desirable from the practical point of view.

Finally, let us look at the solution of the Bézout identity over R_{QM} which is closely related to the existence of a Bézout factorization. As stated above, if a pair $A(s), B(s) \in R_{QM}$ is Bézout coprime, it is possible to find a solution of the Bézout identity or, equivalently, to find the $\operatorname{GCD}\big(A(s), B(s)\big)$ by means of the extended Euclidean algorithm (see subsection 2.2).

Define the poset $P = (R_{QM}, \preceq)$ for $A(s), B(s) \in R_{QM}$ as follows: $A(s) \preceq B(s)$ iff $A(s)|B(s)$; $A(s) \equiv B(s)$ iff $A(s)|B(s)$ and $B(s)|A(s)$, or equivalently, $A(s)$ is associated with $B(s)$; $A(s)$ is not related to $B(s)$ iff $A(s) \nmid B(s)$ and $B(s) \nmid A(s)$. Once the poset for R_{QM} is established, the extended Euclidean algorithm (7), solving $A(s)X(s) + B(s)Y(s) = \operatorname{GCD}\big(A(s), B(s)\big)$ for a Bézout coprime pair $A(s), B(s) \in R_{QM}$, can be used. Consider the following three possibilities:

a) If $A(s) \preceq B(s)$, keep the following scheme:

$$\left[\begin{array}{cc|c} 1 & 0 & B(s) \\ 0 & 1 & A(s) \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -\dfrac{B(s)}{A(s)} & 0 \\ 0 & 1 & A(s) \end{array}\right] \quad (15)$$

Iand hence, $A(s) = \operatorname{GCD}\big(A(s), B(s)\big)$. If $A(s) \succeq B(s)$, the procedure is analogous to the result $B(s) = \operatorname{GCD}\big(A(s), B(s)\big)$.

b) If $A(s) \equiv B(s)$, then $\operatorname{GCD}\big(A(s), B(s)\big)$ is simply either $A(s)$ or $B(s)$ (or any term from the ring associated with them).

c) Let $A(s)$ and $B(s)$ be not related to each other. In this case, follow the scheme

$$\begin{aligned}
&\left[\begin{array}{cc|c} 1 & 0 & A(s) \\ 0 & 1 & B(s) \end{array}\right] \\
&\sim \left[\begin{array}{cc|c} X(s) & 0 & A(s)X(s) \\ 0 & 1 & B(s) \end{array}\right] \\
&\sim \left[\begin{array}{cc|c} X(s) & Y(s) & A(s)X(s) + B(s)Y(s) \\ 0 & 1 & B(s) \end{array}\right] \\
&\sim \left[\begin{array}{cc|c} 0 & 1 & B(s) \\ X(s) & Y(s) & A(s)X(s) + B(s)Y(s) \end{array}\right] \\
&\sim \left[\begin{array}{cc|c} \dfrac{-B(s)X(s)}{A(s)X(s) + B(s)Y(s)} & \dfrac{A(s)X(s)}{A(s)X(s) + B(s)Y(s)} & \\ X(s) & Y(s) & \\ & & 0 \\ & & A(s)X(s) + B(s)Y(s) \end{array}\right]
\end{aligned} \quad (16)$$

Here, $\operatorname{GCD}\big(A(s), B(s)\big) = A(s)X(s) + B(s)Y(s)$where it is assumed that there can be found quotients $X(s), Y(s) \in R_{QM}$ such that the element $T(s) = A(s)X(s) + B(s)Y(s)$ divides $A(s)$, $B(s)$. Since $A(s)$, $B(s)$ are Bézout coprime, $T(s)$ must be a unit of the ring. In other words, the objective is to find structures of $X(s)$, $Y(s)$ and to set zeros and poles of $T(s)$ such that divisibility conditions as in Lemma 4.7 are satisfied or the element is invertible. This task can be troublesome because of a possibility of a neutral numerator in $T(s)$. However, a Bézout coprime pair $A(s)$, $B(s)$ has only a finite number of unstable zeros, which would make it possible to find the $\operatorname{GCD}\big(A(s), B(s)\big)$.

Example 4.2. Assume Bézout coprime factorization (14) and find $\operatorname{GCD}\big(A(s), B(s)\big)$. Since $A(s)$ divides $B(s)$, it holds that $B(s) \succeq A(s)$, hence according to (15) we have $\operatorname{GCD}\big(A(s), B(s)\big) = A(s) = (s-1)/(s-1) = 1$.

Example 4.3. Let the system and its coprime factorization be

$$\begin{aligned}
G(s) &= \frac{s + \exp(-s)}{s^2 + \big(-1 + \exp(-s)\big)s + 1} = \frac{b(s)}{a(s)} \\
&= \frac{\dfrac{s + \exp(-s)}{(s+1)^2}}{\dfrac{s^2 + \big(-1 + \exp(-s)\big)s + 1}{(s+1)^2}} = \frac{\dfrac{b(s)}{m(s)}}{\dfrac{a(s)}{m(s)}} = \frac{B(s)}{A(s)}.
\end{aligned} \quad (17)$$

In this case, $A(s) \nmid B(s)$ and $B(s) \nmid A(s)$, and thus, both elements are not related to one another since $a(s)$ has a complex conjugate root in $\mathbb{C}_0^+$. Following scheme (16) yields a possible calculation:

$$\begin{aligned}
&X(s) = 1,\ Y(s) = 5 \\
&\quad \Rightarrow A(s)X(s) + B(s)Y(s) \\
&\quad = \frac{s^2 + \big(4 + \exp(-s)\big)s + 1 + 5\exp(-s)}{(s+1)^2} \\
&\quad = \operatorname{GCD}\big(A(s), B(s)\big),
\end{aligned}$$

where $X(s)$, $Y(s)$ are chosen as real constants for the simplicity.

Then, for instance, a particular solution of the Bézout identity (5) by using (8) reads

$$\tilde{X}_0(s) = \frac{(s+1)^2}{s^2 + (4+\exp(-s))s + 1 + 5\exp(-s)}$$

$$\tilde{Y}_0(s) = \frac{5(s+1)^2}{s^2 + (4+\exp(-s))s + 1 + 5\exp(-s)}$$

with the parameterization

$$\begin{aligned}\tilde{X}(s) &= \frac{(s+1)^2}{s^2 + [4+\exp(-s)]s + 1 + 5\exp(-s)} \\ &\pm \frac{s+\exp(-s)}{(s+1)^2} T(s) \\ \tilde{Y}(s) &= \frac{5(s+1)^2}{s^2 + [4+\exp(-s)]s + 1 + 5\exp(-s)} \\ &\mp \frac{s^2 + [-1+\exp(-s)]s + 1}{(s+1)^2} T(s)\end{aligned} \tag{18}$$

for any $T(s) \in R_{QM}$.

5. Controller design in R_{QM} – an example

The following simple example concisely demonstrates the utilization of the herein analyzed ring R_{QM} to control design for TDS.

Example 5.1. Consider the habitual simple negative feedback control system, in which $r(t)$ stands for the reference, $e(t)$ is the control error, and $o(t)$ represents the system output. Let the plant be governed by the Bézout coprimely factorized transfer function (17). Due to Proposition 2.2, the control system is stable (in R_{QM} sense) if and only if the Bézout identity (5) holds true where $C(s) = Y(s)/X(s)$ stands for the controller transfer function (the proof can be made analogously to [21]). In Example 4.3, all stabilizable controllers are parameterized by (18) with $\tilde{X}(s) \neq 0$. In the further text, we take $X(s)$, $Y(s)$ rather than $\tilde{X}(s)$, $\tilde{Y}(s)$ for simplicity reasons.

Now the question is how to set $T(s) \in R_{QM}$ in order to meet other control performance requirements. In practice, a common task is to track the reference signal $r(t) = \mathcal{L}^{-1}\{r(s)\} = \mathcal{L}^{-1}\{H_r(s)/F_r(s)\}$. Since (if (5) holds true) the reference-to-error transfer function reads $G_{re}(s) = E(s)/R(s) = A(s)X(s)$, $r(t)$ is asymptotically tracked by the output $o(t)$ if $F_r(s)$ divides $A(s)X(s)$, i.e. when $A(s)X(s)/F_r(s)\} \in R_{QM}$. Therefore,

$$\begin{aligned}&\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sG_{re}(s)E(s) \\ &= \lim_{s\to 0} sA(s)X(s)H_r(s)/F_r(s) = 0.\end{aligned} \tag{19}$$

Let the reference be from the family of step-wise functions, i.e. $F_r(s) = s$. In order to have $X(s)$ in a sufficiently simple form, choose

$$T(s) = t_0 \frac{(s+1)^2}{s^2 + (4+\exp(-s))s + 1 + 5\exp(-s)}, \; t_0 \in \mathbb{R}.$$

By using simple algebra, the condition $X(0) = 0$ that agrees with (19), yields $t_0 = 1$. The substitution into (18) results in

$$X(s) = \frac{s^2 + s + 1 - \exp(-s)}{s^2 + (4+\exp(-s))s + 1 + 5\exp(-s)},$$

$$Y(s) = \frac{6s^2 + (11 - \exp(-s))s + 6}{s^2 + (4+\exp(-s))s + 1 + 5\exp(-s)}.$$

Thus, the eventual infinite-dimensional controller reads

$$C(s) = \frac{6s^2 + (11 - \exp(-s))s + 6}{s^2 + s + 1 - \exp(-s)}. \tag{20}$$

The step response of the feedback system can be seen in Fig. 1. Note that the overall performance can be improved either by the selection of other values of $X(s)$, $Y(s)$ (see Example 4.3) or their degrees, or by the introduction of more degrees of freedom in $T(s)$.

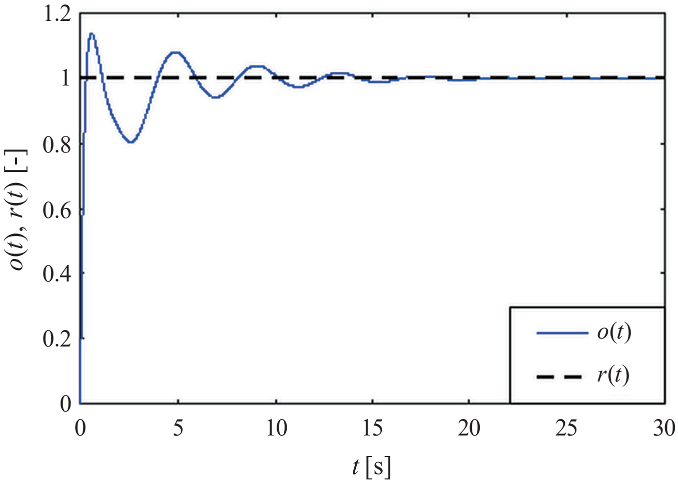

Fig. 1. Feedback step response with plant (17) and controller (20)

6. Conclusions

To sum up, the concept of the ring of stable quasipolynomial meromorphic functions, R_{MS}, for TDS has been attacked and extended giving rise to the new ring R_{QM} covering neutral and distributed delays. It has been shown that the original conception has some crucial deficiencies, mainly from the algebraic point of view, and hence, that it should be revised. We have then introduced basic algebraic and functional properties of R_{QM}, presented as lemmas that are mostly proved. For the engineering practice, some algebraic operations over the ring have been discussed as well, and control design affairs of the novel proposition are touched by means of a concise illustrative example. Throughout the paper, many examples are presented to illuminate some ideas and results. However, there are many relevant topics that still remain open, for instance to derive other algebraic properties, inclusions, and relationships between some other algebras.

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Appendix.

Proof of Proposition 3.1. Let $T_1(s) = n_1(s)/d_1(s)$, $T_2(s) = n_2(s)/d_2(s)$, where $n_1(s)$, $d_1(s)$, $n_2(s)$, $d_2(s)$ are quasipolynomials as in (2). Define the sets of their zeros as $\sum_{n,1}$, $\sum_{d,1}$, $\sum_{d,1}$, $\sum_{d,2}$, respectively. Then, the zeros and poles of $T_i(s)$ agree with elements of sets $\sum_{Z,i} = \sum_{n,i} \backslash \left(\sum_{n,i} \cap \sum_{d,i}\right)$, $\sum_{P,i} = \sum_{d,i} \backslash \left(\sum_{n,i} \cap \sum_{d,i}\right)$, respectively. In addition, let us introduce subsets $\sum_{\cdot,i}^{+}$, those with $\mathrm{Re}(\sigma_{ik}) \geq 0$. Since both terms are formally stable, the number of unstable poles is finite, i.e. $\left|\sum_{P,i}^{+}\right| < \infty$. Now consider operations of addition and multiplication $T_A(s) = T_1(s) + T_2(s)$ and $T_M(s) = T_1(s)T_2(s)$. Again, both resultants have a finite number of unstable poles, since the poles of $T_A(s)$ are entries of the set $\sum_{A,P} = \sum_{P,1} \cup \sum_{P,2}$ except for those that are zeros of $T_A(s)$, and the set of all poles of $T_M(s)$ agrees with $\sum_{M,P} = \left(\sum_{P,1} \backslash \left(\sum_{Z,2} \cap \cap \sum_{P,1}\right)\right) \cup \left(\sum_{P,2} \backslash \left(\sum_{Z,1} \cap \sum_{P,2}\right)\right)$. Again, since $\left|\sum_{P,i}^{+}\right| < \infty$, then $\left|\sum_{M,i}^{+}\right| < \infty$. □

Proof of Lemma 4.1. It is sufficient to verify ring properties and the commutativity of multiplication over R_{QM}. Obviously, R_{QM} is closed under addition, since it is known that if $T_1(s)$, $T_2(s) \in H_\infty$, then $T_1(s) + T_2(s) \in H_\infty$, and according to Proposition 2.3, formal stability is not affected by the operation. Associativity is evident, the neutral element for addition is simply $E_A = 0$ and inverse element $I_{A,T}(s) = R_{QM}$ of $T(s) \in R_{QM}$ reads $I_{A,T}(s) = -T(s)$. The closure under multiplication with associativity is also clear because of $T_1(s)T_2(s) \in H_\infty$ and Proposition 3.1. In the case of distributed delays, it is not possible to obtain more unstable denominator zeros than numerator ones of any $T(s) \in R_{QM}$ under multiplication. The multiplicative identity element E_M equals 1. Since the operation of quasipolynomial multiplication is commutative, the ring is commutative, and left and right distributivity hold as well. □

Proof of Lemma 4.3. *Necessity*. Use the indirect proof and consider the following three cases for which (12) does not hold: a) $O_R = 0$, $N_U > 1$; b) $O_R > 1$, $N_U = 0$; c) $O_R > 1$, $N_U > 0$.

For a), consider a (quasi)polynomial $x_U(s)$ with only one unstable real zero (or a single pair of unstable zeros) of $T(s)$, say $x_U(s_{U,1}) = 0$ (or $x_U(s_{U,1}) = x_U(\bar{s}_{U,1}) = 0$), and an arbitrary stable (quasi)polynomial $x_S(s)$ of the same order (i.e. first or second one). Then, one can write

$$T(s) = \frac{t_{num}(s)}{t_{den}(s)} = \frac{t_{num}(s)x_S(s)}{t_{den}(s)x_U(s)}\frac{x_U(s)}{x_S(s)} = T_1(s)T_2(s), \tag{21}$$

where $T_1(s)$, $T_2(s) \in R_{QM}$ are neither associated with $T(s)$ nor units.

Assume b) and a stable (quasi)polynomial $z_S(s)$ of the first order, and follow the scheme

$$T(s) = \frac{t_{num}(s)z_S(s)}{t_{den}(s)}\frac{1}{z_S(s)} = T_3(s)T_4(s). \tag{22}$$

Again, $T_3(s)$, $T_4(s) \in R_{QM}$ are neither associated with $T(s)$ nor units.

Finally, for c) it is possible to adopt e.g. factorization (22).

Sufficiency. Formula (12) admits, in fact, only two possibilities: a) $O_R = 0$, $N_U = 1$ and b) $O_R = 1$, $N_U = 0$.

Our intention is to show that if these conditions hold, it is possible to construct neither (21) nor (22). Considering case a) and the formally stable numerator, scheme (21) fails, since $T_1(s)$ is a unit and $T_2(s)$ is associated with $T(s)$. Moreover, it is not possible to find another "reducible" scheme. Analogously, if the second point holds and $T_{num}(s)$ is formally stable, $T_1(s)$ is a unit and $T_2(s)$ is associated with $T(s)$ in scheme (22), and hence, $T(s)$ is irreducible. □

Proof of Lemma 4.4. In the first step it is easy to see that R_{QM} meets the definition of an integral domain. Indeed, as mentioned in Introduction, it holds that two variables s and $z = \exp(-\tau s)$, where τ is some base delay, are in quasipolynomial (2) algebraically independent over $\mathbb{R}$, i.e. there is no nontrivial linear combination of s, z over $\mathbb{R}$, such that quasipolynomial $q(s)$ in (2) is identically zero. Thus, for any $T_1(s) = n_1(s)/d_1(s) \neq 0$, $T_2(s) = n_2(s)/d_2(s) \neq 0 \in R_{QM}$, i.e. $n_1(s)$, $n_2(s) \neq 0$, it is not possible to obtain $n_1(s)n_2(s) = 0$ in the multiplication $T_1(s)T_2(s)$.

To prove that the R_{QM} ring is not a UFD, consider $T(s) = \left(1 - \exp(-\tau s)\right)/s \in R_{QM}$, the zeros of which (i.e. nonzero roots of its numerator) read $s_k, \bar{s}_k = \pm(2k\pi/\tau)\mathrm{j}$, $k \in \mathbb{N}$. Define a set of polynomials $P_k(s) = (s - s_k)(s - \bar{s}_k)$. Then $T(s)$ can be factorized as

$$\begin{aligned} \frac{1-\exp(-\tau s)}{s} &= \frac{(1-\exp(-\tau s))(s+m_0)^2}{sP_1(s)}\frac{P_1(s)}{(s+m_0)^2} = \\ &= \frac{(1-\exp(-\tau s))(s+m_0)^4}{sP_1(s)P_2(s)}\frac{P_1(s)P_2(s)}{(s+m_0)^4} = \ldots, \end{aligned} \tag{23}$$

where $m_0 > 0$. The chain of successive factorizations is infinite, none of left-hand factors in (23) is irreducible (see Lemma 4.3) for $k < \infty$, none of all factors is a unit (compare with Lemma 4.2) or associated with $T(s)$ and thus, the R_{QM} ring is not a UFR. □

Proof of Lemma 4.8. *Necessity*. Assume a contradiction. If $\deg_s m(s) < \deg_s a(s)$, then $A(s)$ is not proper. On the contrary, the choice $\deg_s m(s) > \deg_s a(s)$ implies that there exists a non-associated, nonzero, nonunit $T(s) \in R_{QM}$ such that $A(s) = \widetilde{A}(s)T(s)$, $B(s) = \widetilde{B}(s)T(s)$, i.e. both terms are prime. Similarly, if there exists a common unstable root of $a(s)$, $b(s)$, say $s_0 \in \mathbb{R}^+$ for the simplicity, which is not included in $m(s)$, it is possible to write

$$\frac{a(s)}{m(s)}=\frac{a(s)}{(s-s_0)\widetilde{m}(s)}\frac{(s-s_0)\widetilde{m}(s)}{m(s)}=\widetilde{A}(s)T(s)$$
$$\frac{b(s)}{m(s)}=\frac{b(s)}{(s-s_0)\widetilde{m}(s)}\frac{(s-s_0)\widetilde{m}(s)}{m(s)}=\widetilde{B}(s)T(s) \tag{24}$$

with a stable (quasi)polynomial $\widetilde{m}(s)$ and noninvertible $T(s) \in R_{QM}$.

Sufficiency. Let us proceed with an indirect proof and consider a prime pair $A(s)$, $B(s) \in R_{QM}$, i.e. that there exists a non-unit $T(s) \in R_{QM}$ satisfying $A(s)=\widetilde{A}(s)T(s)$, $B(s)=\widetilde{B}(s)T(s)$ for some $\widetilde{A}(s)$, $\widetilde{B}(s) \in R_{QM}$. According to Lemma 4.2, $T(s)$ has a positive relative order, or at least one unstable zero or its numerator is formally unstable. If the relative order is positive, it means that $A(s)$, $B(s)$ are strictly proper, i.e. $\deg_s m(s) > \deg_s a(s)$. If there exists an unstable zero of $T(s)$, it must be a common zero of both $A(s)$, $B(s)$. However, we have a contradiction, since $m(s)$ should cancel all such common zeros and it cannot be included in $\widetilde{A}(s)$, $\widetilde{B}(s)$ because of their stability. Finally, the existence of a common, formally unstable numerator (factor) would yield a formally unstable quasipolynomial $m(s)$ having common unstable zeros, which is unfeasible due to Definition 2.3. □

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